

Disorder induced localization and enhancement of entanglement in one- and two-dimensional quantum walks

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The time evolution of one- and two-dimensional discrete-time quantum walk with increase in disorder is studied. We use spatial, temporal and spatio-temporal broken periodicity of the unitary evolution as disorder to mimic the effect of disordered/random medium in our study. Disorder induces a dramatic change in the interference pattern leading to localization of the quantum walks in one- and two-dimensions. Spatial disorder results in the decreases of the particle and position entanglement in one-dimension and counter intuitively, an enhancement in entanglement with temporal and spatio-temporal disorder is seen. The study signifies that the Anderson localization of quantum state without compromising on the degree of entanglement could be implement in a large variety of physical settings where quantum walks has been realized. The study presented here could make it feasible to explore, theoretically and experimentally the interplay between disorder and entanglement. This also brings up a variety of intriguing question relating to the negative and positive implications on algorithmic and other applications.

Introduction- Localization of all energy states of a single particle in disordered potential is one of the remarkable predictions of quantum mechanics. It was originally predicted in the context of transport of electors in disordered crystals by Anderson¹. This phenomenon originates from the broken periodicity in the dynamics of the systems due to disordered media leading the interference between multiple scattering paths to result in the absence of diffusion, localization². It has been experimentally observed and theoretically studied in a variety of systems³, including light waves⁴ and matter waves⁵. Quantum walks which are the quantum analog of classical random walk also evolve involving the interference of amplitudes of multiple traversing paths⁶⁻⁹. Converse to interference leading to localization, interference in quantum walks results in a quadratically faster spread in position space when compared to its classical counterpart in one-dimension^{10,11}. Some studies have also shown the interference of amplitudes between multiple traversing paths in the quantum walk resulting in the localization around the origin from various different perspectives¹²⁻¹⁸. In particular, localization of the walk dynamics in one- dimension is shown by introducing drifts with constant momentum between two consecutive steps of the quantum walk¹² or by evolving the walk in a random medium characterized by a static disorder¹⁵.

Interference leading to faster diffusion of quantum walks have played a significant role in the development of quantum algorithms¹⁹. Subsequently, it is widely used to simulate and explain dynamics in variety of the physical systems. For example, they have been used to explain the mechanism of wavelike energy transfer within photosynthetic systems^{20,21}, to demonstrate the coherent quantum control over atoms^{22,23} and to explore topological phases²⁴. Experimental implementation of quantum walk in NMR system, ions, photons, and atoms²⁵ have opened up a new dimension to simulate quantum dynamics in physical systems like the recent demonstration of localization of photon's wavepacket²⁶. In an an-

ticipation of continuation of significant role of quantum walks in understanding and simulating dynamics in various naturally occurring and engineered systems in laboratory, its relevant to have a effect of disorder on the dynamics. Quantum walks has been studied in two forms, continuous-time quantum walk (CTQW)²⁷ and discrete-time quantum walk (DTQW)⁸⁻¹¹. In this letter we will study the effect of disorder in DTQW.

Unlike the classical random walk, the DTQW using the standard form of evolution is deterministic and has no randomness associated with it. Disorder introduces randomness into the system and have shown the deviation from quadratic speed-up to the localization of the interference of amplitudes of the multiple traversing paths of the walk around the origin^{12-14,16,17,23}. The key factor for the interference effect to result in localization is the broken periodicity in the dynamics of the system induced by the disordered media. Broken periodicity need not be mediated by a disordered or a random medium alone; operations defining the dynamics of the system can be made random to break the periodicity in the evolution such that they mimic the effect of a random medium in the system and manifest localization^{14,23}. Taking this into consideration, we introduce disorder into the one- and two-dimensional DTQW system using randomized unitary evolution and call it as a *randomized DTQW*. Unitary preserving randomized DTQW evolution operator is modelled in different forms: spatial disorder, temporal disorder and spatio-temporal disorder by assigning different quantum coin operation for each position, time, and both position-time, respectively. Analytically, we show that the disorder induced in the form of randomized unitary evolution affects the interference pattern leading to localization of DTQW in one- and two-dimensions. Our study is further supported by the numerical analysis of the dynamics by comparing the standard deviation of the probability distribution of the disordered evolution. In addition we also present the effect of disorder on the degree of entanglement between the particle and

the position space. We show that the spatial disorder results in the decreases of the particle-position entanglement and counter intuitively, an enhancement in entanglement with temporal and spatio-temporal disorder is seen. This signifies that the localization of quantum state without compromising on the degree of entanglement can be realized in a large variety of physical settings where DTQW has been realized. However localization might have a negative effect on algorithmic applications which require faster spatial spread but can serve as a strong positive sign where localization of quantum states is required. In addition, randomized unitary operations can assist localization photons in photonic systems which can find applications in photon storage and quantum information processing.

Discrete-time quantum walks - DTQW in one-dimension is modelled using the two-state particle on a position space. The Hilbert space of the particle \mathcal{H}_c is defined by the internal states of the particle, $|\uparrow\rangle = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $|\downarrow\rangle = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ as the basis state and the position Hilbert space \mathcal{H}_p is defined by the basis state described in terms of $|x\rangle$, where $x \in \mathbb{I}$. Each step of DTQW on the Hilbert space $\mathcal{H} = \mathcal{H}_c \otimes \mathcal{H}_p$ comprises of the quantum coin operation, $B(\theta) \equiv \begin{bmatrix} \cos(\theta) & \sin(\theta) \\ -\sin(\theta) & \cos(\theta) \end{bmatrix}$ to evolve the particle in superposition of the basis states of the particle followed by the unitary shift operation $S \equiv \sum_x [|\uparrow\rangle\langle\uparrow| \otimes |x-1\rangle\langle x| + |\downarrow\rangle\langle\downarrow| \otimes |x+1\rangle\langle x|]$ which shift the states $|\uparrow\rangle$ and $|\downarrow\rangle$ of the particle to the left and right of the position space in superposition, respectively. Therefore, the effective operation for each step of the DTQW is written in the form:

$$W_x(\theta) \equiv S_x[B(\theta) \otimes \mathbb{1}] \quad (1)$$

and the state after the t step, $|\Psi_t\rangle = W(\theta)^t |\Psi_{\text{in}}\rangle$, where $|\Psi_{\text{in}}\rangle = (\cos(\delta/2)|\uparrow\rangle + e^{i\eta}\sin(\delta/2)|\downarrow\rangle) \otimes |0\rangle$, is the initial state of the particle at a position $x = 0$. The coin parameter θ controls the variance of the probability distribution in position space. The state of the particle at each position x and time $t + 1$ (after $t + 1$ step) in the form of left- and right- moving components is,

$$\psi_L(x, t+1) = \cos(\theta)\psi_L(x+1, t) + \sin(\theta)\psi_R(x-1, t) \quad (2a)$$

$$\psi_R(x, t+1) = \cos(\theta)\psi_R(x-1, t) - \sin(\theta)\psi_L(x+1, t). \quad (2b)$$

The left- moving (ψ_L) and right- moving (ψ_R) components can be decoupled and written as

$$\psi(x, t+1) + \psi(x, t-1) = \cos(\theta) [\psi(x-1, t) + \psi(x+1, t)]. \quad (3)$$

The decoupled form of the left- and right- moving component are identical to each other therefore, we can write $\psi = \psi_L = \psi_R$. Subtracting both side of the Eq. (3) by

$2[1 + \cos(\theta)]\psi(x, t)$ and changing the difference form of the expression to second order differential equation,

$$\left[\frac{\partial^2}{\partial t^2} - \cos(\theta) \frac{\partial^2}{\partial x^2} + 2[1 - \cos(\theta)] \right] \psi(x, t) = 0. \quad (4)$$

The above expression can be rewritten in the Klein-Gordon equation form

$$\left[\frac{1}{\xi_N} \frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x^2} + 2\zeta_N \right] \Psi(x, t) = 0, \quad (5)$$

showing the free spin-0 particle relativistic character of each component of the DTQW²⁸. From Eq. (4) the equivalent to speed of light of each component ψ_L and ψ_R is ξ_N , $c \equiv \sqrt{\xi_N}$. This shows that for each time (step) of the DTQW the displacement $D = \sqrt{\cos(\theta)}$ ($\theta = \pi/4$ for standard evolution). Therefore, after time t , the left- and the right moving component of the particle initially at $x = 0$ will travel a distance of $D(t) = t\sqrt{\cos(\theta)}$ from the origin and this agree with the earlier studies. We should note that $\cos(\theta)$ will be negative for any $\theta \in \{\pi/2, \pi\}$. This should be interpreted as the displacement of the left moving component to the right and the right moving component to the left. Therefore, for any fixed θ for all time t , the effective displacement will be $D(t) = |t\sqrt{\cos(\theta)}|$.

Disorder evolution and localization- We introduce disorder into the DTQW system using randomized unitary evolution in different forms:

(a) *Spatial disorder* : By making the coin operation, a position dependent operation with different value of $\theta_x \in \{0, \pi\}$ randomly chosen for each position x , spatial disorder can be introduced to the DTQW evolution. The state after t step with spatial disorder will be

$$|\Psi_t\rangle_S = \left[W_x(\tilde{\theta}_x) \right]^t |\Psi_{\text{in}}\rangle = S_x[B(\tilde{\theta}_x) \otimes \mathbb{1}]^t |\Psi_{\text{in}}\rangle, \quad (6)$$

where $B(\tilde{\theta}_x)$ is the position dependent coin operation. Since the coin operation is not uniform for the entire position space we need to consider the differential equation form similar to Eq. (4) for each position x to obtain the distance travelled by each step of the walk. Therefore, the general expression $\forall x$ will be

$$\sum_{x=-t}^t \left[\frac{\partial^2}{\partial t^2} - \cos(\theta_x) \frac{\partial^2}{\partial x^2} + 2[1 - \cos(\theta_x)] \right] \psi(x, t) = 0, \quad (7)$$

$$\text{which is equivalent to } \left[\frac{\partial^2}{\partial t^2} - \xi_{SD} \frac{\partial^2}{\partial X^2} + 2\zeta_{SD} \right] \Psi(X, t) = 0,$$

where $\Psi(X, t) = \sum_{x=-t}^t \psi(x, t)$. Therefore, the displacement for each time t ,

$$D_{SD} = \sqrt{\xi_{SD}} = \sqrt{\sum_{x=-t}^{x=t} \frac{\cos(\theta_x)}{(2t+1)}}. \quad (8)$$

In the above expression for small time t the distribution of θ_x from the full range $\{0, \pi\}$ will be very narrow resulting in a small displacement. For higher value of t , $\sqrt{\sum_{x=-t}^{x=t} \cos(\theta_x)} \approx 0$, bringing the displacement to halt. In addition, depending of the values of θ_x , D_{SD} can be a real number or a complex number corresponding to the displacement in the positive direction or the negative direction, respectively. Therefore, adding up the displacement for each time t ,

$$\sum_{t=1}^t \left[\sqrt{\sum_{x=-t}^{x=t} \frac{\cos(\theta_x)}{2t+1}} \right] = D_{SD}^+(t) + iD_{SD}^-(t), \quad (9)$$

where $D_{SD}^+(t)$ and $D_{SD}^-(t)$ is the total displacement in the positive and negative directions, respectively. From this the effective displacement after time t will be

$$D_{SD}(t) = |D_{SD}^+(t) - D_{SD}^-(t)| \approx 0. \quad (10)$$

Thus, the spatial disorder in DTQW induces a strong localization of the particle in position space. It should also be noted that a disordered operations on a small fraction of position space results only in a small displacements with time compared to the standard evolution (weak localization regime).

(b) *Temporal disorder* : Randomly assigning different coin operation, that is, $\theta_t \in \{0, \pi\}$ for each step of the DTQW evolution introduces temporal disorder into the system^{14,23}. Therefore, the state after t step with temporal disorder will be

$$|\Psi_t\rangle_T = W_x(\theta_t) \dots W_x(\theta_3)W_x(\theta_2)W_x(\theta_1)|\Psi_{in}\rangle. \quad (11)$$

For each position x at time t the differential equation form will be similar to Eq. (4) with only a replacement of $\cos(\theta)$ by $\cos(\theta_t)$,

$$\left[\frac{\partial^2}{\partial t^2} - \cos(\theta_t) \frac{\partial^2}{\partial x^2} + 2[1 - \cos(\theta_t)] \right] \psi(x, t) = 0. \quad (12)$$

The displacement for each time t will be $D_t = \sqrt{\cos(\theta_t)}$. Adding the displacement from each step with temporal disorder we get,

$$\sum_{t=1}^t \sqrt{\cos(\theta_t)} = D_{TD}^+(t) + iD_{TD}^-(t). \quad (13)$$

With random θ for each time t will be effective results in the displacement of

$$D_{TD}(t) = \text{abs}[D_{TD}^+(t) - D_{TD}^-(t)] \approx 0. \quad (14)$$

From the preceding analysis we can conclude that with increase in time t the effective displacement will tend towards zero, inducing localization of the particle in position space.

(c) *Spatio-Temporal disorder* - Randomly assigning different coin operation for each position (spatial) and changing that configuration for each step (temporal) will induce a spatio-temporal disorder or fluctuating disorder in the DTQW evolution. The state after t step with spatio-temporal disorder will be

$$\begin{aligned} |\Psi_t\rangle_{S-T} &= W_x(\tilde{\theta}_{x,t}) \dots W_x(\tilde{\theta}_{x,3})W_x(\tilde{\theta}_{x,2})W_x(\tilde{\theta}_{x,1})|\Psi_{in}\rangle \\ &= S_x[B(\tilde{\theta}_{x,t}) \otimes \mathbb{1}] \dots S_x[B(\tilde{\theta}_{x,1}) \otimes \mathbb{1}]|\Psi_{in}\rangle, \end{aligned} \quad (15)$$

where $B(\tilde{\theta}_{x,t})$ is the position and time dependent coin operation. The expression to describe the dynamics at time t will be same as Eq. 7 and changes for each time, that is θ_x will be replaced by $\theta_{x,t}$. The displacement for each time t will be,

$$D_{S-TD} = \sqrt{\sum_{x=-t}^{x=t} \frac{\cos(\theta_{x,t})}{(2t+1)}}. \quad (16)$$

Adding up the displacement from each time t ,

$$\sum_{t=1}^t \left[\sqrt{\sum_{x=-t}^{x=t} \frac{\cos(\theta_{x,t})}{2t+1}} \right] = D_{S-TD}^+(t) + iD_{S-TD}^-(t), \quad (17)$$

From this the effective displacement after some time t will be

$$D_{S-TD}(t) = |D_{S-TD}^+(t) - D_{S-TD}^-(t)| \approx 0, \quad (18)$$

inducing localization of DTQW with spatio-temporal disordered operations.

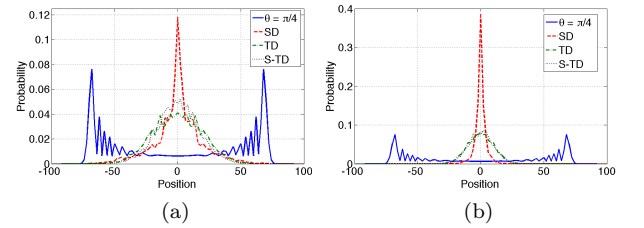


FIG. 1: (Color online) Probability distribution after 100 steps of 1D standard quantum walk evolution ($\theta = \pi/4$) and for evolutions with spatial, temporal and spatio-temporal disordered coin operations after 100 steps when the percentage of disorder in (a) and (b) is 20% and 100%, respectively. Spatial disorder shows a relatively stronger localization effect compared to the temporal and spatio-temporal disorder.

In Fig.1, we show the probability distribution after 100 steps of the standard DTQW evolution ($\theta = \pi/4$) and the evolution with spatial, temporal and spatio-temporal disordered operations. In Fig.1(a), only 20% of the coin operations (θ_x , θ_t , and $\theta_{x,t}$) are randomly chosen for the evolution with remaining 80% of the operations with $\theta = \pi/4$. Even a small fraction of disorder shows the

trend towards localization this can be seen as a weak localization regime. In Fig. 1(b), the coin operations (θ_x , θ_t , and $\theta_{x,t}$) are randomly chosen for the complete evolution. Spatial disorder has a relatively stronger effect on the diffusion compared to the temporal and spatio-temporal disorder.

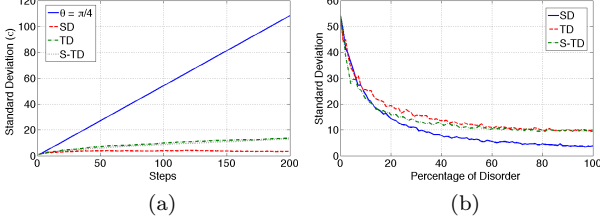


FIG. 2: (Color online) (a) Standard deviation as a function of step (t) for normal evolution with $\theta = \pi/4$ and for evolutions with spatial, temporal and spatio-temporal disordered coin operations. (b) Standard deviation as a function of percentage of disordered operation starting from the normal evolution with $\theta = \pi/4$ after 100 steps. For evolution with disorder, standard deviation does not show any noticeable increment with increase in steps showing the signature of localization and a significant decrease in standard deviation is seen with increase in percentage of disorder.

In Fig. 2, we show the standard deviation obtained numerically for the DTQW with standard ($\theta = \pi/4$), spatial, temporal and spatio-temporal evolutions. Fig. 2(a), shows a significant increase in standard deviation as a function of steps (t) for standard DTQW evolution but for evolution with disorder, absence of noticeable increment with disorder shows the signature of localization. In Fig. 2(b), standard deviation after 100 steps of walk as a function of percentage of disorder is shown. We can see a sudden decrease in standard deviation even for small increase in percentage of disorder. In all this, effect due to spatial disorder is stronger compared to the temporal and spatio-temporal disorder.

Disorder and effect on entanglement - Entanglement, which is a very useful resource in quantum information processing is generated during the DTQW evolution. In an entangled system, noise or disorder results in decrease or death of entanglement but in our study the disorder introduced does not affect the unitarity of the evolution. Therefore, its effect on entanglement is not significantly large. In Fig 3 we show the effect of disorder on the degree of entanglement measured using von Neumann entropy, $E(\rho) = -\text{Tr}(\rho_c \log \rho_c)$, where ρ_c is the density matrix of the DTQW system after tracing out the position space. In Fig. 3(a), degree of entanglement as a function of time is shown and it is directly proportional to the size of the position space in which the particle is found in superposition. Because of strong localization induced by spatial disorder the sites in which particle is seen in superposition is relatively less when compared to the DTQW with $\theta = \pi/4$ resulting in the decrease of entanglement.

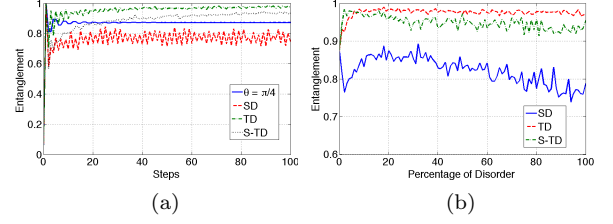


FIG. 3: (Color online) (a) Degree of entanglement as a function of steps and (b) as a function of percentage of disorder (after 100 step) using von Neumann entropy as a measure for standard evolution of quantum walk and evolution with spatial, temporal and spatio-temporal disordered coin operations. Spatial disorder suppresses the degree of entanglement whereas, temporal and spatio-temporal disorder enhances the entanglement.

Though $\theta = \pi/4$ spreads the distribution wide in position space, but the amplitude of the particle is largely concentrated on a small number of position space which are at the extreme points in the position space. Therefore, even with localization around the origin the spread of amplitude in superposition of position space with temporal and spatio-temporal disorder is large enhancing the entanglement. In Fig 3(b), we show the degree of entanglement with increase in percentage of disorder and a similar effect of decrease in entanglement with spatial disorder and enhancement is temporal and spatio-temporal disorder is seen.

Localization in two-dimension - Two-state DTQW on a square lattice can be realized by evolving the particle in one axis followed by the evolution in the other axis²⁹. Alternatively, a two state walk on square lattice can be constructed using different Pauli basis states as translational states for the two axis^{28,30} and we will use this scheme and study the effect of disorder in two-dimension. For convenience we will choose the eigenstates of the Pauli operator $\hat{\sigma}_3 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$, $|\uparrow\rangle$ and $|\downarrow\rangle$

as basis states for x -axis and eigenstates of $\hat{\sigma}_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, $|+\rangle = \frac{1}{\sqrt{2}}(|\uparrow\rangle + |\downarrow\rangle)$ and $|-\rangle = \frac{1}{\sqrt{2}}(|\uparrow\rangle - |\downarrow\rangle)$ as basis states for y -axis. In this scheme a coin operation is not necessary and each step of the walk compose of $S_{0,y}S_{x,0}$, where $S_{x,0} \equiv \sum_{x,y} (|\uparrow\rangle\langle\uparrow| \otimes |x-1, y\rangle\langle x, y| + |\downarrow\rangle\langle\downarrow| \otimes |x+1, y\rangle\langle x, y|)$ and $S_{y,0} \equiv \sum_{x,y} (|+\rangle\langle+| \otimes |x, y-1\rangle\langle x, y| + |-\rangle\langle-| \otimes |x, y+1\rangle\langle x, y|)$.

To introduce disorder it is necessary to introduce coin operation therefore, we will introduce coin operation after evolution in each direction and the state after t step will be,

$$|\Psi_t^{2d}\rangle = [S_{0,y}(B_y(\theta) \otimes \mathbb{1})S_{x,0}(B_x(\theta) \otimes \mathbb{1})]^t |\Psi_{in}\rangle, \quad (19)$$

where $B_x(\theta) = B(\theta)$ and $B_y(\theta) = \cos(\theta)|+\rangle\langle+| + \sin(\theta)|+\rangle\langle-| - \sin(\theta)|-\rangle\langle+| + \cos(\theta)|-\rangle\langle-|$. In the same

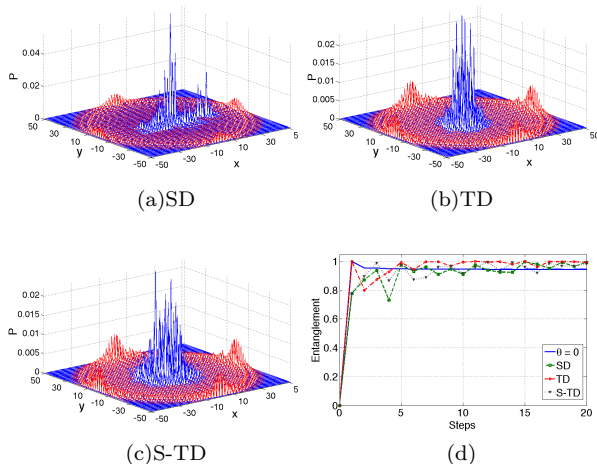


FIG. 4: (Color online) Comparison of the Probability distribution after 50 steps of 2D standard quantum walk evolution with (a) spatial disorder, (b) temporal, and (c) spatio-temporal disorder coin operations. (d) Degree of entanglement as a function of steps using von Neumann entropy as a measure for standard evolution of quantum walk and evolution with spatial, temporal and spatio-temporal disordered coin operations. We can see the localization of distribution for disordered evolutions and a small enhancement of entanglement is seen for disordered evolution.

way as we introduced disorder into the DTQW on a line we can introduce spatial, temporal and spatio-temporal disorder, by randomly choosing different $\theta \in \{0, \pi\}$ for each position, time and both position-time, respectively.

The numerical evolution on a square lattice using different form of disordered evolution operation is shown in

Fig. 4. Although interference effect is seen in the distribution, we note that the localization of the probability distribution around the origin with all the three forms of disorder. Disorder on square lattice has a very small effect on the degree of entanglement when compared to the effect it had on the one dimensional DTQW system. However small it is, we can still note the enhancement of entanglement with all the forms of disorder in two-dimension.

In conclusion, disorder induces localization of DTQW which normally diffuse quadratically faster when compared to the classical random walk. Interestingly, the interference during the evolution is what plays a significant role for faster diffusion in normal DTQW evolution and for localization due to disordered evolution. Our study modelled using spatial, temporal and spatio-temporal disordered unitary operations in the form of coin operations had a noticeable effect on the degree of entanglement between the particle and position space. Interesting, disorder in general enhances the entanglement in both, one- and two-dimension except for spatial disorder in one-dimension where a small suppression of entanglement is seen. This opens up an important question for future, interplay between the disorder and entanglement. Anderson localization is a universal phenomenon in addition, it can also enhance the entanglement leaving open, both the negative and positive implications on its applications to algorithmic and to understand the dynamics in various other quantum systems. Randomized unitary operations can assist localization (storage) of quantum state in variety of physical systems and applications in quantum information processing. This also give more options to further explore topological phase, and understand physical process in nature.

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